By hypothesis f_n' exists for each point of (a,b) (7)

$\Rightarrow f_n'(c)$ exists

$\Rightarrow$ ie $f_n'(c) = \lim_{x \to c} \dfrac{f_n(x) - f_n(c)}{x - c}$ exists

$\Rightarrow f_n'(c) = \lim_{x \to c} g_n(x)$ by (1)

ie) $g_n(c) = \lim_{x \to c} g_n(x) \quad \because g_n(c) = f_n'(c)$

$\Rightarrow \lim_{x \to c} g_n(x) = g_n(c)$

$\therefore g_n$ is continuous at c for each n.

ie) each g_n is continuous at c

Let $G(x) = \lim_{n \to \infty} g_n(x) \quad \forall x \in (a,b)$

Then $\{g_n\}$ converges uniformly to G on (a,b). Since each g_n is continuous at c and since $g_n \to G$ uniformly on (a,b) the limit function G is also continuous at c (by thm 2)

$$\Rightarrow \lim_{x \to c} G(x) = G(c) \longrightarrow (9)$$

But for $x \neq c$ we have

$$G(x) = \lim_{n \to \infty} g_n(x)$$

$$\Rightarrow G(x) = \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c}$$

$$= \frac{\lim_{n \to \infty} f_n(x) - \lim_{n \to \infty} f_n(c)}{x - c}$$

$$G(x) = \frac{f(x) - f(c)}{x - c} \longrightarrow (10)$$

Now (10) implies that,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} G(x)$$

$f'(c)$ exists and equals $G(c)$ ie) $f'(c) = G(c) \longrightarrow (11)$

But $G(c) = \lim_{n \to \infty} g_n(c)$

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But $G(c) = \lim_{n \to \infty} g_n(c)$

$= \lim_{n \to \infty} f_n'(c)$ [by ①]

$= g(c)$ [$f_n' \to g$ uniformly on (a,b)]

$\therefore G(c) = g(c)$

From ① & ②

$f'(c) = g(c)$

Since c is an arbitrary point of (a,b),

$f'(x) = g(x) \quad \forall x \in (a,b)$

Theorem: 9.14

Assume that each f_n is a real valued function defined on (a,b) such that the derivative $f_n'(x)$ exists for each x in (a,b). Assume that, for atleast one point x_0 in (a,b) the series $\sum f_n(x_0)$ converges. Assume further that there exists a function g such that

$\sum f_n'(x) = g(x)$ [uniformly on (a,b)]

then. a) There exists a function f such that

$\sum f_n(x) = f(x)$ [uniformly on (a,b)]

b) If $x \in (a,b)$, the derivative $f'(x)$ exists and equals $\sum f_n'(x)$

Proof:

Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) \quad \forall n$

Since $f_n'(x)$ exists for each $x \in (a,b)$

$s_n'(x)$ exists for each $x \in (a,b)$ and $\longrightarrow ①$

$s_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$

Since there exists a function g such that

$\sum f_n'(x) = g(x)$ uniformly on (a,b)

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ie $S_n' \to g$ uniformly on (a,b) $\longrightarrow$ (2)

Since it is given that for atleast one point $x_0 \in (a,b)$ the series $\sum b_n(x_0)$ converges.

$\therefore$ The sequence of partial sums $\{s_n(x_0)\}$ converges $\longrightarrow$ (3)

Here all the conditions of theorem 9.13 are satisfied, (from equations (1), (2), and (3))

Therefore (i) There exists a function f such that the sequence of partial sum $s_n \longrightarrow f$ uniformly on (a,b)

$\Rightarrow \sum b_n(x) = f(x)$ uniformly on (a,b) and (ii) for each $x \in (a,b)$ the derivative $f'(x)$ exists $f'(x) = g(x)$

$\Rightarrow f'(x) = \sum b_n'(x)$ $\quad [\because g(x) = \sum b_n'(x)]$

Hence the proof.

Sufficient Conditions for Uniform Convergence of a Series

Theorem: 9.15 Dirichlet's Test for Uniform Convergence:

V Q 6m

Statement:

Let $F_n(x)$ denote the n^{th} partial sum of the series $\sum b_n(x)$ where each b_n is a complex valued function defined on a set S.

Assume that $\{F_n\}$ is uniformly bounded on S.

Let $\{g_n\}$ be a sequence of real valued functions such that $g_{n+1}(x) \leq g_n(x)$ for each x in S and for every $n = 1, 2 \ldots$ and assume that $g_n \to 0$ uniformly on S.

Then the series $\sum b_n(x)\, g_n(x)$ converges uniformly on S.

proof:

Given:

(i) $F_n(x)$ denote the n^{th} partial sum of the Series $\sum f_n(x)$, where each f_n is a complex valued function defined on a set S.

ie) $F_n(x) = \sum_{k=1}^{n} f_k(x) \longrightarrow (1)$

(ii) $\{F_n\}$ is uniformly bounded on S. then there exists a Constant $M>0$ such that,

$|F_n(x)| \le M \quad \forall \quad x \in S$ and $\forall$ n. $\longrightarrow (2)$

(iii) $\{g_n\}$ is a sequence of real-valued functions Such that

$g_{n+1}(x) \le g_n(x)$ for each $x \in S$ & $\forall$ $n=1,2 \dots$ $\longrightarrow (3)$

(iv) $g_n \to 0$ uniformly on S.

Then for every $\epsilon>0$, there exists N Such that

$|g_n(x) - 0| < \epsilon/2M$ for every $x \in S$ and $n>N$

$|g_n(x)| < \epsilon/2M \quad \forall \quad x \in S$ and $n>N \longrightarrow (4)$

T.P.T: The Series $\sum f_n(x)\, g_n(x)$ converges uniformly on S.

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)\, g_k(x)$

T.P.T: $\sum f_n(x) . g_n(x)$ converges.

we have T.P.T $\{S_n\}$ satisfies cauchy condition for uniformly convergence.

(ie) TPT for every $\epsilon>0$ $\exists$ an integer $N \ni$

$|S_n(x) - S_m(x)| < \epsilon \quad \forall$ $n>N$, $m>N$ and $\forall x \in S$

By partial summation we have,

$S_n(x) = \sum_{k=1}^{n} f_k(x)\, g_k(x)$

$$= \sum_{k=1}^{n} [F_k(x) - F_{k-1}(x)]\, g_k(x)$$

$$= \sum_{k=1}^{n} F_k(x)\, g_k(x) - \sum_{k=1}^{n} F_{k-1}(x)\, g_k(x)$$

$$= \sum_{k=1}^{n} F_k(x)\, g_k'(x) - \sum_{k=2}^{n} F_{k-1}(x)\, g_k(x) \quad [\text{By taking } F_0(x) = 0]$$

$$= \sum_{k=1}^{n} F_k(x)\, g_k(x) - \sum_{k=1}^{n-1} F_k(x)\, g_{k+1}(x)$$

$$= \sum_{k=1}^{n} F_k(x)\, g_k(x) - \sum_{k=1}^{n} F_k(x)\, g_{k+1}(x) + F_n(x)\, g_{n+1}(x) \longrightarrow (5)$$

Illy

$$s_m(x) = \sum_{k=1}^{n} F_k(x)\,[g_k(x) - g_{k+1}(x)] + F_m(x)\, g_{m+1}(x)$$

If $n > m$ we have,

$$s_n(x) - s_m(x) = \sum_{k=1}^{n} F_k(x)\,[g_k(x) - g_{k+1}(x)] + F_n(x)\, g_{n+1}(x) - \sum_{k=1}^{m} F_k(x)\,[g_k(x) - g_{k+1}(x) - F_m(x)\, g_{m+1}(x)]$$

$$\Rightarrow s_n(x) - s_m(x) = \sum_{k=m+1}^{n} F_k(x)\,[g_k(x) - g_{k+1}(x)] + F_n(x)\, g_{n+1}(x) - F_m(x)\, g_{m+1}(x)$$

$$|s_n(x) - s_m(x)| \le \sum_{k=m+1}^{n} |F_k(x)|\,|g_k(x) - g_{k+1}(x)| + |F_n(x)|\,|g_{n+1}(x)| + |F_m(x)|\,|g_{m+1}(x)|$$

$$= \sum_{k=m+1}^{n} |F_k(x)|\,(g_k(x) - g_{k+1}(x)) + |F_n(x)|\,(g_{n+1}(x)) + |F_n(x)|\,(g_{m+1}(x))$$

$$\le \sum_{k=m+1}^{n} M\,(g_k(x) - g_{k+1}(x)) + M\,(g_{n+1}(x)) + M\,(g_{m+1}(x))$$

$$= M\{g_{m+1}(x) - g_{m+2}(x) + g_{n+2}(x) - g_{m+3}(x) + \cdots + g_n(x) - g_{n+1}(x)\} + Mg_{n+1}(x) + Mg_{m+1}(x)$$

$$= M\{g_{m+1}(x) - g_{n+1}(x)\} + Mg_{m+1}(x) + Mg_{m+1}(x)$$

$$= 2M\, g_{m+1}(x)$$

$$\Rightarrow |\omega_n(x) - \omega_m(x)| \le 2M\, g_{m+1}(x) \quad \text{---} (6)$$

(4) $\Rightarrow$ $|g_n(x)| < \epsilon/2M$, $n > N$ and $\forall\ x \in S$

But $g_n(x) \ge 0$

$g_n(x) < \epsilon/2M$, $n > N$ and $\forall\ x \in S$

Using in (6) we have,

$$|\omega_n(x) - \omega_m(x)| < 2M \cdot \frac{\epsilon}{2M}$$

Thus for every $\epsilon > 0$, there exists an N such that

$|\omega_n(x) - \omega_m(x)| < \epsilon$, $\forall$ $n > N$, $m > N$ and $\forall\ x \in S$

$\therefore$ $\{\omega_n\}$ satisfies cauchy condition for uniform convergence of sequences of function

$\therefore$ $\{\omega_n\}$ converges uniformly on S

$\therefore$ The corresponding series $\sum_{n=1}^{\infty} f_n(x)\, g_n(x)$ converges uniformly on S.

Abel's Test for Uniform Convergence:

Let $\{g_n\}$ be a sequence of real valued function such that $g_{n+1}(x) \le g_n(x)$ for each x in S and for every $n = 1, 2, \ldots$ If $\{g_n\}$ is uniformly bounded on S and if $\sum f_n(x)$ converges uniformly on S, then $\sum f_n(x)\, g_n(x)$ also converges uniformly on S

(OR)

$\sum f_n(x)\, g_n(x)$ converges uniformly on S if

(i) $\sum f_n(x)$ converges uniformly on S

(ii) $\{g_n(x)\}$ is a decreasing sequence which is uniformly bounded on S.

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proof:

Given: (i) $\sum b_n(x)$ converges uniformly on S

Then by Cauchy condition for uniform convergence of Series we have,

for any given $\epsilon>0$, there is an N suchthat,

$$\left|\sum_{k=n+1}^{n+p} b_k(x)\right| < \frac{\epsilon}{3M} \quad \text{for each } p=1,2\ldots \text{ and } n>N$$

and for every $x \in S$ → ①

Let $_pR_n(x) = b_{n+1}(x) + b_{n+2}(x) + \ldots + b_{n+p}(x)$

Then (1) ⇒ when $p=1$, $|{}_1R_n(x)| < \frac{\epsilon}{3M}$ $\forall x \in S$ $\forall n>N$

when $p=2$, $|{}_2R_n(x)| < \frac{\epsilon}{3M}$ $\forall x \in S$ $\forall n>N$

In general for $n>N$ we have,

$$|{}_pR_n(x)| < \frac{\epsilon}{3M} \quad \forall x \in S \ \& \ \forall p=1,2\ldots$$

→ ②

given: (ii) $\{g_n(x)\}$ is a decreasing sequence on S

then $g_{n+1}(x) \le g_n(x)$ $\forall n$ and $\forall x \in S$

→ ③

given: (iii) $\{g_n(x)\}$ is uniformly bounded on S

then there exists a constant $M>0$ such that

$|g_n(x)| \le M$ $\forall x \in S$ and $\forall n$.

→ ④

T.P.T $\sum_{n=1}^{\infty} b_n(x) g_n(x)$ converges uniformly on S

we have T.P.T

for any given $\epsilon>0$ there is an integer N such that

$$\left|\sum_{k=n+1}^{n+p} b_k(x) g_k(x)\right| < \epsilon, \quad \forall n>N, \ p=1,2\ldots \ \forall x \in S$$

Consider,

$$\sum_{k=n+1}^{n+p} b_k(x) g_k(x) = b_{n+1}(x) g_{n+1}(x) + b_{n+2}(x) g_{n+2}(x) + \ldots + b_{n+p-1}(x) g_{n+p-1}(x) + b_{n+p}(x) g_{n+p}(x)$$

$$= {}_1R_n(x)\, g_{n+1}(x) + \left({}_2R_n(x) - {}_1R_n(x)\right) g_{n+2}(x) + \cdots +$$

$$\left({}_{p-1}R_n(x) - {}_{p-2}R_n(x)\right) g_{n+p-1}(x) + \left({}_pR_n(x) - {}_{p-1}R_n(x)\right) g_{n+p}(x)$$

$$= {}_1R_n(x)\left(g_{n+1}(x) - g_{n+2}(x)\right) + {}_2R_n(x)\left(g_{n+2}(x) - g_{n+3}(x)\right)$$

$$+ {}_{p-1}R_n(x)\left(g_{n+p-1}(x) - g_{n+p}(x)\right) + {}_pR_n(x)\, g_{n+p}(x)$$

$$\Rightarrow \left|\sum_{k=n+1}^{n+p} f_k(x)\, g_k(x)\right|$$

$$\leq \left|{}_1R_n(x)\right| \left|g_{n+1}(x) - g_{n+2}(x)\right| + \left|{}_2R_n(x)\right| \left|g_{n+2}(x) - g_{n+3}(x)\right|$$

$$+ \cdots + \left|{}_{p-1}R_n(x)\right| \left|g_{n+p-1}(x) - g_{n+p}(x)\right| + \left|{}_pR_n(x)\right| \left|g_{n+p}(x)\right|$$

$$< \frac{\epsilon}{3M}\left|g_{n+1}(x) - g_{n+2}(x)\right| + \frac{\epsilon}{3M}\left|g_{n+2}(x) - g_{n+3}(x)\right| + \cdots +$$

$$\frac{\epsilon}{3M}\left|g_{n+p-1}(x) - g_{n+p}(x)\right| + \frac{\epsilon}{3M}\left|g_{n+p}(x)\right|$$

$$= \frac{\epsilon}{3M}\left(g_{n+1}(x) - g_{n+2}(x)\right) + \frac{\epsilon}{3M}\left(g_{n+2}(x) - g_{n+3}(x)\right)$$

$$+ \cdots + \frac{\epsilon}{3M}\left(g_{n+p-1}(x) - g_{n+p}(x)\right) + \frac{\epsilon}{3M}\left|g_{n+p}(x)\right|$$

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\, g_k(x)\right| < \frac{\epsilon}{3M}\left\{g_{n+1}(x) - g_{n+2}(x) + g_{n+2}(x) - g_{n+3}(x)\right.$$

$$\left. + \cdots + g_{n+p-1}(x) - g_{n+p}(x)\right\} + \frac{\epsilon}{3M}\left|g_{n+p}(x)\right|$$

$$= \frac{\epsilon}{3M}\left\{g_{n+1}(x) - g_{n+p}(x)\right\} + \frac{\epsilon}{3M}\left|g_{n+p}(x)\right|$$

$$= \frac{\epsilon}{3M}\left|g_{n+1}(x) - g_{n+p}(x)\right| + \frac{\epsilon}{3M}\left|g_{n+p}(x)\right|$$

$$\leq \frac{\epsilon}{3M}\left\{\left|g_{n+1}(x)\right| + \left|g_{n+p}(x)\right|\right\} + \frac{\epsilon}{3M}\left|g_{n+p}(x)\right|$$

$$= \frac{\epsilon}{3M}\left\{\left|g_{n+1}(x)\right| + 2\left|g_{n+p}(x)\right|\right\}$$

$$< \frac{\epsilon}{3M}\left\{M + 2M\right\} = \frac{\epsilon}{3M}\cdot 3M = \epsilon$$

$$\Rightarrow \left|\sum_{k=n+1}^{n+p} f_k(x)\, g_k(x)\right| < \epsilon \quad \forall\, n > N,\ p = 1, 2, \ldots \ \&\ \forall\, x \in S$$

$\therefore$ $\sum f_n(x)\, g_n(x)$ converges uniformly on S.

Example :

Establish the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$$

Soln : Take $b_n(x) = e^{inx}$ and

$$g_n(x) = 1/n$$

Then (i) The sequence $\{g_n\}$ is such that

$$g_{n+1}(x) \leq g_n(x) \text{ for each } x \in R \text{ and } n = 1, 2, \ldots 3 \longrightarrow (1)$$

(ii) $g_n \to 0$ uniformly on $R \longrightarrow (2)$

(iii) Let $F_n(x) = \sum_{k=1}^{n} b_k(x) \quad , x \in R$

$$= \sum_{k=1}^{n} e^{ikx} \quad , x \in R$$

T.P.T $\{F_n\}$ is uniformly bounded.

Consider $\sum_{k=1}^{n} e^{ikx} = F_n(x)$

By the theorem, for every real $x \neq 2m\pi$ (m is an integer), we have.

$$\sum_{k=1}^{n} e^{ikx} = e^{ix}\left(\frac{1-e^{inx}}{1-e^{ix}}\right)$$

$$= \frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot e^{i(n+1)x/2}$$

$$|F_n(x)| = \left|\frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right| \left|e^{i(n+1)x/2}\right|$$

$$\leq \frac{1}{\sin(x/2)} \quad \text{for } x \neq 2m\pi \text{ (} m \text{ is an integer)}$$

Therefore if $0 < \delta < \pi$ we have,

$$|F_n(x)| \leq \frac{1}{\sin\left(\frac{\delta}{2}\right)} \quad \text{if } \delta \leq x \leq 2\pi - \delta$$

Hence $\{F_n\}$ is uniformly bounded on the interval $[\delta, 2\pi - \delta] \longrightarrow (3)$

Applying Dirichlet's test for uniform convergence we find that the series

$$\sum_{n=1}^{\infty} f_n(x)\, g_n(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{n}$$ converges uniformly on $[\delta, 2\pi-\delta]$ if $0<\delta<\pi$

9.13 Mean Convergence

Def: 9.17

Let $\{f_n\}$ be a sequence of Riemann integrable functions defined on $[a,b]$. Assume that $f \in R$ on $[a,b]$. The sequence $\{f_n\}$ is said to converge in the mean to f on $[a,b]$, and we write

$$\lim_{n\to\infty} f_n = f \text{ on } [a,b],$$

$$\text{if } \lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2\, dx = 0.$$

Note: 1 Uniform Convergence implies mean Convergence:

A Sequence $\{f_n\}$ converges uniformly to f on $[a,b]$, then $\{f_n\}$ converges also in the mean to f on $[a,b]$

Suppose $f_n \to f$ uniformly on $[a,b]$ then for any given $\epsilon>0$ we can find N Such that,

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in [a,b] \text{ and } n>N$$

Hence for $n>N$ we have,

$$\int_a^b |f_n(x) - f(x)|^2\, dx < \epsilon \int_a^b f\, dx$$

$$= \epsilon^2 (b-a)$$

$$\Rightarrow \int_a^b | f_n(x) - f(x)|^2\, dx < \epsilon^2 (b-a)$$

$$\Rightarrow \left| \int_a^b |f_n(x) - f(x)|^2 dx - 0 \right| < \epsilon^2 (b-a)$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b | f_n(x) - f(x)|^2\, dx = 0$$

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This proves that $\lim_{n\to\infty} f_n = f$ provided that each f_n is Riemann-integrable on $[a,b]$

Note : 2

Convergence of a sequence of some closed interval in the mean need not imply pointwise convergence at any point of the interval.

For each integer $n \geq 0$, subdivide the interval $[0,1]$ into 2^n equal subintervals,

Let I_{2^n+k} denote the subinterval whose right end point is $\frac{k+1}{2^n}$, where $k = 0, 1, 2, \dots 2^n - 1$

This yields a collection $\{I_1, I_2, \dots\}$ of subintervals of $[0,1]$ which the first few are

$$I_1 = [0,1], \quad I_2 = [0, 1/2]$$
$$I_3 = [1/2, 1], \quad I_4 = [0, 1/4]$$
$$I_5 = [1/4, 1/2], \quad I_6 = [1/2, 3/4]$$

and define f_n on $[0,1]$ as follows

$$f_n(x) = \begin{cases} 1 & \text{if } x \in I_n \\ 0 & \text{if } x \in [0,1] - I_n \end{cases}$$

constructed converges in the mean to $f \equiv 0$ on $[0,1]$

indeed.

$$\int_0^1 |f_n(x) - 0|^2 dx = \int_{I_n} f_n^2(x)\, dx$$
$$= \int_{I_n} dx$$
$$= \text{length of } I_n \longrightarrow 0 \text{ as } n \to \infty$$

At the same time the sequence $\{f_n(x)\}$ does not converge for any x in $[0,1]$,

Since for each $x \in [0,1]$ we have

$\lim_{n\to\infty} \sup f_n(x) = 1$ and

$\lim_{n\to\infty} \inf f_n(x) = 0$.

Theorem 9.18

Assume that $\lim_{n \to \infty} f_n = f$ on $[a,b]$. If $g \in R$ on $[a,b]$ define

$$h(x) = \int_a^x f(t) g(t)\, dt, \quad h_n(x) = \int_a^x f_n(t) g(t)\, dt$$

if $x \in [a,b]$, then $h_n \to h$ uniformly on $[a,b]$.

Proof: given.

i) $\lim_{n \to \infty} f_n = f$ on $[a,b]$

(ii) $g \in R$ on $[a,b]$

(iii) $h(x) = \int_a^x f(t) g(t)\, dt$

$$h_n(x) = \int_a^x f_n(t)\, g(t)\, dt \quad \text{if } x \in [a,b]$$

T.P.T $h_n \to h$ uniformly on $[a,b]$

ie) T.P.T for any given $\epsilon > 0$ there exists an integer N (depending only on ϵ) such that and $\forall x \in [a,b]$

$$|h_n(x) - h(x)| < \epsilon \quad \forall n \geq N$$

consider,

$$h_n(x) - h(x) = \int_a^x f_n(t) g(t)\, dt - \int_a^x f(t) g(t)\, dt$$

$$h_n(x) - h(x) = \int_a^x \big(f_n(t) - f(t)\big) g(t)\, dt$$

$$|h_n(x) - h(x)| \leq \int_a^x |f_n(t) - f(t)|\,|g(t)|\, dt \longrightarrow (1)$$

By Cauchy - Schwartz inequality,

If $f \in R(\alpha)$ and $g \in R(\alpha)$, $f^2, g^2 \in R(\alpha)$ on $[a,b]$

we have,

$$\left(\int_a^b f(x) g(x)\, d\alpha(x) \right)^2 \leq \left(\int_a^b (f(x))^2\, d\alpha(x) \right) \left(\int_a^b (g(x))^2\, d\alpha(x) \right)$$

Since $f_n \in R$ on $[a,b]$ and $f \in R$ on $[a,b]$

$f - f_n \in R$ on $[a,b]$

$\therefore |f-f_n|^2 \in R$ on $[a,b]$

Such that $g \in R$ on $[a,b]$

consider,

$$0 \leq \left(\int_a^x |f(t) - f_n(t)| \, |g(t)| \, dt \right)^2$$

$$\leq \left(\int_a^x |f(t) - f_n(t)|^2 dt \right) \left(\int_a^x |g(t)|^2 dt \right) \longrightarrow (2)$$

Given that,

$$\lim_{n \to \infty} f_n = f$$

then by the definition of mean convergence we have,

$$\lim_{n \to \infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0$$

$\therefore$ For any given $\epsilon > 0$ there exists N such that

$$\left| \int_a^b |f_n(t) - f(t)|^2 dt - 0 \right| < \epsilon^2 / A \quad \forall \ n > N.$$

$$\Rightarrow \int_a^b |f_n(t) - f(t)|^2 dt < \epsilon^2 / A \longrightarrow (3)$$

where $A = 1 + \int_a^b |g(t)|^2 dt$

$$A - 1 = \int_a^b |g(t)|^2 dt.$$

Using (3) in (2) we have,

$$0 \leq \left(\int_a^x |f(t) - f_n(t)| \, |g(t)| \, dt \right)^2$$

$$\leq \left(\int_a^x |f(t) - f_n(t)|^2 dt \right) \left(\int_a^x |g(t)|^2 dt \right)$$

$$< \frac{\epsilon^2}{A} (A-1) \qquad \left\{ \because A - 1 = \int_a^b |g(t)^2| dt \text{ and } \frac{A-1}{A} < 1 \right\}$$

$$< \epsilon^2$$

$$\Rightarrow 0 \leq \left(\int_a^x |f(t) - f_n(t)| |g(t)| \, dt \right)^2 < \epsilon^2$$

$$\Rightarrow \int_a^x |f(t) - f_n(t)| \, |g(t)| \, dt < \epsilon \longrightarrow (4)$$

Sub (4) in (1) we get

$$(1) \Rightarrow |h_n(x) - h(x)| \leq \int_a^x |f_n(t) - f(t)| \, |g(t)| \, dt$$

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$< \epsilon$

$\Rightarrow |h_n(x) - h(x)| < \epsilon \quad \forall \; x \in [a,b] \;\&\; \forall \; n \geq N.$

$\Rightarrow h_n \to h$ uniformly on $[a,b]$

Theorem: 9.19

P Assume that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ on $[a,b]$

Define, $h(x) = \int_a^x f(t)g(t)\,dt, \quad h_n(x) = \int_a^x f_n(t)\,g_n(t)\,dt$ if

$x \in [a,b]$, then $h_n \to h$ uniformly on $[a,b]$

Proof: Given, (i) $\lim_{n\to\infty} f_n = f$

(ii) $\lim_{n\to\infty} g_n = g$

(iii) $h(x) = \int_a^x f(t)\,g(t)\,dt$ and

$h_n(x) = \int_a^x f_n(t)\,g_n(t)\,dt$ if $x \in [a,b]$

TP: $h_n \to h$ uniforsformly on $[a,b]$.

Let $x \in [a,b]$

consider,

$$h_n(x) - h(x) = \int_a^x f_n(t)\,g_n(t)dt - \int_a^x f(t)\,g(t)\,dt$$

$$= \int_a^x [(f_n g_n)(t) - (fg)(t)]\,dt$$

$$= \int_a^x (f_n g_n - fg)(t)\,dt$$

Sub and Add the three product $f \cdot g$, $f_n g$ and $f g_n$

$$= \int_a^x [f_n g_n + fg - fg + f_n g - f_n g + f g_n - f g_n - fg]\,dt$$

$$= \int_a^x (f - f_n)(g - g_n)\,dt + \int_a^x (f_n g - fg)\,dt + \int_a^x (g_n f - gf)\,dt \quad \to (1)$$

(Recall the cauchy schwarz inequality

then replace g by $g - g_n$)

Using cauchy - ~~Lagrange~~ schwarz inequality we get,

$$\left(\int_a^x |(f-f_n)(g-g_n)|\,dt\right)^2 \le \left(\left|\int_a^x (f-f_n)\,dt\right|\right)^2 \left(\left|\int_a^x (g-g_n)\,dt\right|\right)^2$$

$$\le \int_a^x |f-f_n|^2\,dt \cdot \int_a^x |g-g_n|^2\,dt \longrightarrow (2)$$

hypothesis : $\lim_{n\to\infty} f_n = f$

$$\int |f_n - f|^2\,dt = 0$$

(or) $$\int |f-f_n|^2\,dt = 0$$

Similarly $\lim_{n\to\infty} g_n = g$ on $[a,b]$ implies

$$\therefore \int |g-g_n|^2\,dt = 0$$

$$(2) \Rightarrow \int_a^x |(f-f_n)(g-g_n)|\,dt < 0 \longrightarrow (3)$$

then $(1) \Rightarrow |h_n(x) - h(x)| = 0 + \int_a^x |f_n g - fg|\,dt + \int_a^x |g_n f - gf|\,dt \to (4)$

Consider, $\int_a^x |f_n g - fg|\,dt = \int_a^x |f_n - f|\,|g|\,dt$

$$\le \int_a^x |(f_n - f)(t)|\,|g(t)|\,dt = 0 \longrightarrow (5)$$

hypothesis : $\lim_{n\to\infty} g_n = g$ on $[a,b]$

$$\therefore \int_a^x |(g_n - g)(t)|\,|f(t)|\,dt = 0 \longrightarrow (6)$$

Sub (5) & (6) in (4)

$$|h_n(x) - h(x)| = 0 < \epsilon, \ \forall x \in [a,b]$$

$\therefore \{h_n\} \to h$ uniformly on $[a,b]$

Hence the theorem.